Data Structures and Algorithms

(CS210A)

Semester I – **2014-15**

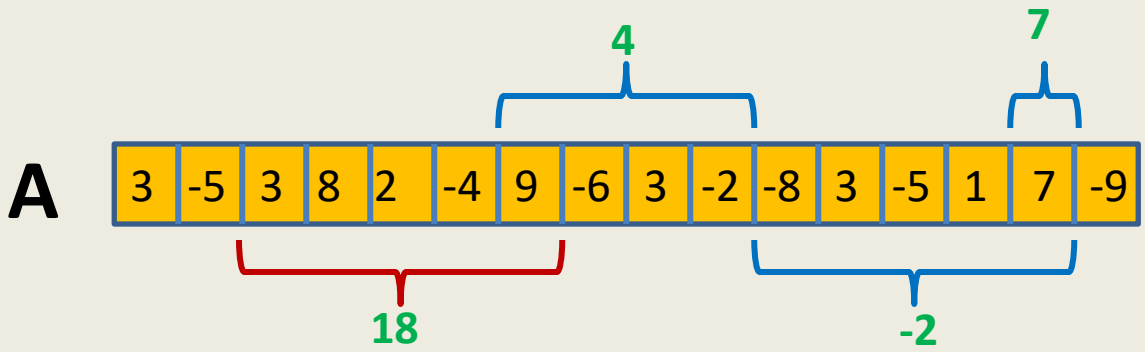
Lecture 4:

- Design of **O**(n) time algorithm for **Maximum sum subarray**
- Proof of **correctness** of an algorithm
- A new problem : **Local Minima in a grid**

Max-sum subarray problem

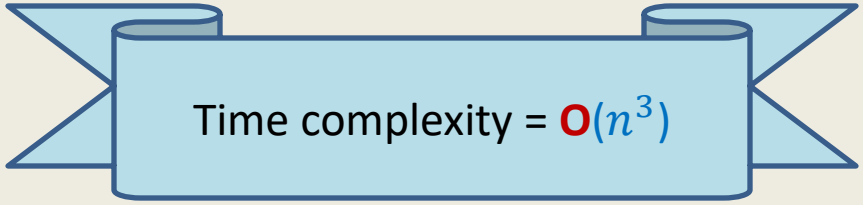
Given an array **A** storing n numbers,

find its **subarray** the sum of whose elements is maximum.



Max-sum subarray problem: A trivial algorithm

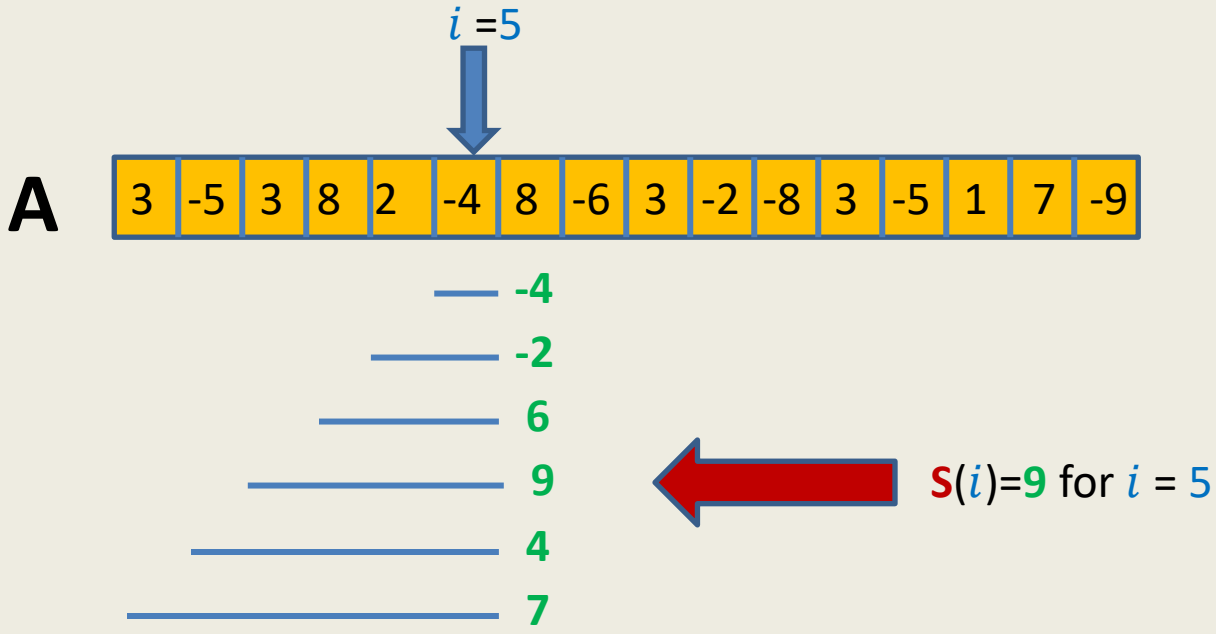
```
A_trivial_algo(A)
{ max← A[0];
  For i=0 to n-1
     For j=i to n-1
          {    temp ← compute_sum(A,i,j);
               if max< temp  then max← temp;
          }
  return max;
}

compute_sum(A, i,j)
{  sum←A[i];
   For k=i+1 to j     sum← sum+A[k];
   return sum;
}
```



DESIGNING AN O(n) TIME ALGORITHM

Focusing on any particular index i

Let **S**(i): the sum of the **maximum-sum subarray ending at index** i.



Observation:

In order to solve the problem, it suffices to compute **S**(i) for each $0 \leq i < n$.

Focusing on any particular index i

Observation:

In order to solve the problem, it suffices to compute **S**(i) for each $0 \leq i < n$.

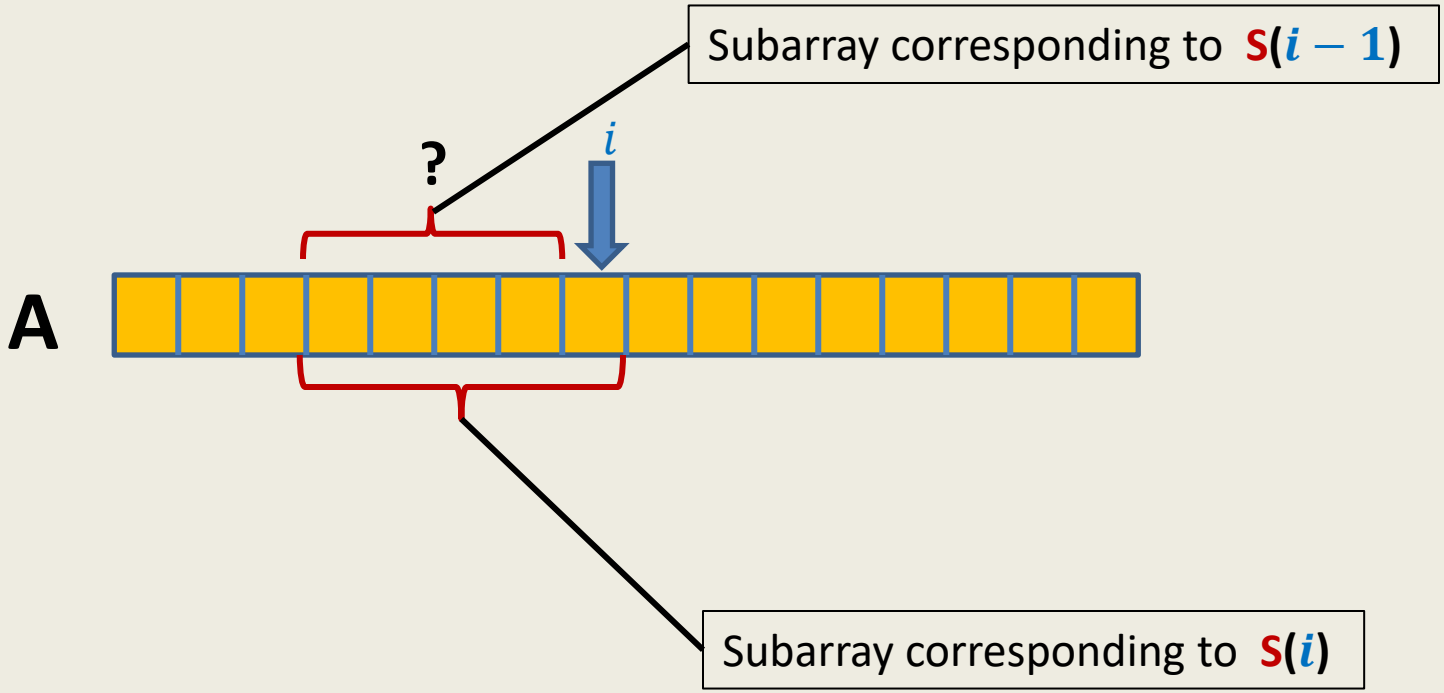
Question: If we wish to achieve O(n) time to solve the problem, how quickly should we be able to compute **S**(i) for a given index i ?

Answer: O(1) time.

Idea: Perhaps we can compute **S**(i) if we know **S**($i-1$) ?

Question: What is the **relation** between **S**(i) and **S**($i-1$) ?

Relation between S(i) and S($i-1$)



Theorem 1:

If S($i-1$) > 0 **then** S(i) = S($i-1$) + A[i]

else S(i) = A[i]

An O(n) time Algorithm for Max-sum subarray

Max-sum-subarray-algo(A[0 ... $n-1$])

{ **S**[0] ← **A**[0];

for i = **1** to $n-1$

{ **If S**[$i-1$] > 0 **then S**[i] ← **S**[$i-1$] + **A**[i]

else S[i] ← **A**[i]

}

"Scan **S** to return the maximum entry"

}

O(**1**) time

$n-1$ **repetitions**

O(**1**) time

O(n) time

Time complexity of the algorithm = **O**(n)

Homework:

- Refine the algorithm so that it uses only **O**(1) extra space.

An O(n) time Algorithm for Max-sum subarray

Max-sum-subarray-algo(A[0 ... $n-1$])

{ **S**[0] ← **A**[0]

for i = **1** to $n-1$

{ **If** **S**[$i-1$] > 0 **then** **S**[i] ← **S**[$i-1$] + **A**[i]

else **S**[i] ← **A**[i]

}

"Scan **S** to return the maximum entry"

}

What is the proof of correctness of the algorithm ?

What does correctness of an algorithm mean ?

For every possible **valid input**, the algorithm must output **correct** answer.

An O(n) time Algorithm for Max-sum subarray

Max-sum-subarray-algo(**A**[0 ... $n-1$])

{ **S**[0] ← **A**[0]

for i = **1** to $n-1$

{ **If S**[$i-1$] > 0 **then S**[i] ← **S**[$i-1$] + **A**[i]

else S[i] ← **A**[i]

}

"Scan **S** to return the maximum entry"

}

Question:

What needs to be proved in order to establish the correctness of this algorithm ?

Answer: At the end of ith iteration,

"**S**[i] stores the sum of maximum sum subarray ending at index i"

Proof of correctness of Max-sum-subarray-algo

Assertion:

At the end of iteration i, **S**[i] stores the sum of maximum sum subarray ending at **A**[i].

Question: How to prove the assertion ?

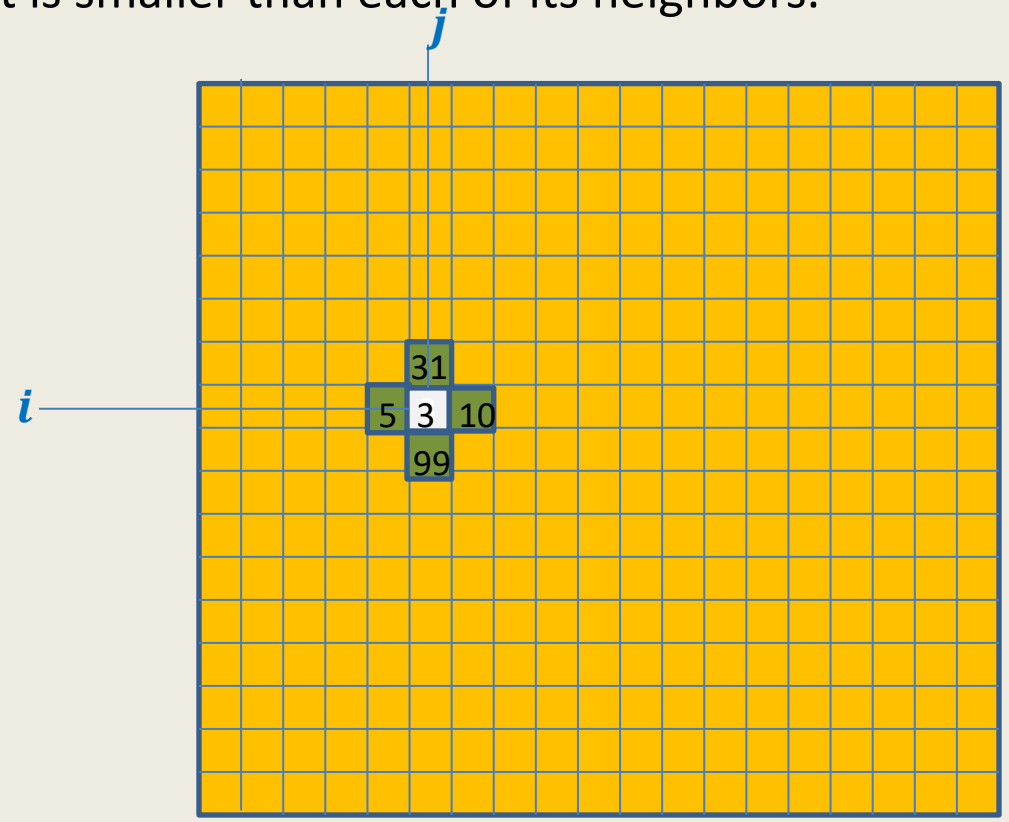
Answer: [By **mathematical induction** and using **Theorem 1**]

Homework: Make sincere attempts to write the details of the proof. (it is quite easy).

NEW PROBLEM:
LOCAL MINIMA IN A GRID

Local minima in a grid

Definition: Given a $n \times n$ grid storing distinct numbers, an entry is local minima if it is smaller than each of its neighbors.

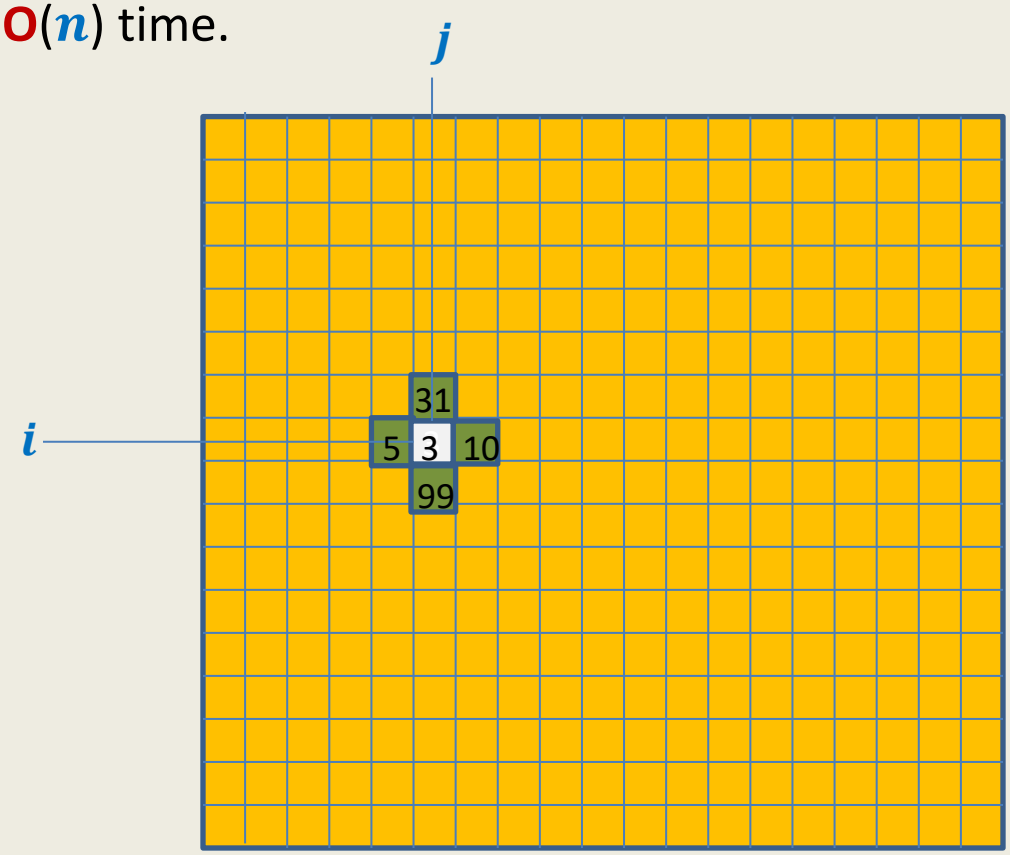


Does a local minima exist always ?

Yes. After all, global minima is also a local minima.

Local minima in a grid

Problem: Given a $n \times n$ grid storing distinct numbers, output any local minima in $O(n)$ time.



Using common sense principles

- There are some simple but very fundamental principles which are not restricted/confined to a specific stream of science/philosophy.
- These principles, which we usually learn as common sense, can be used in so many diverse areas of human life.
- For the current problem of local minima, we shall use two such simple principles.

Two simple principles

1. Respect every new idea which solves a problem even partially.

2. Principle of simplification:

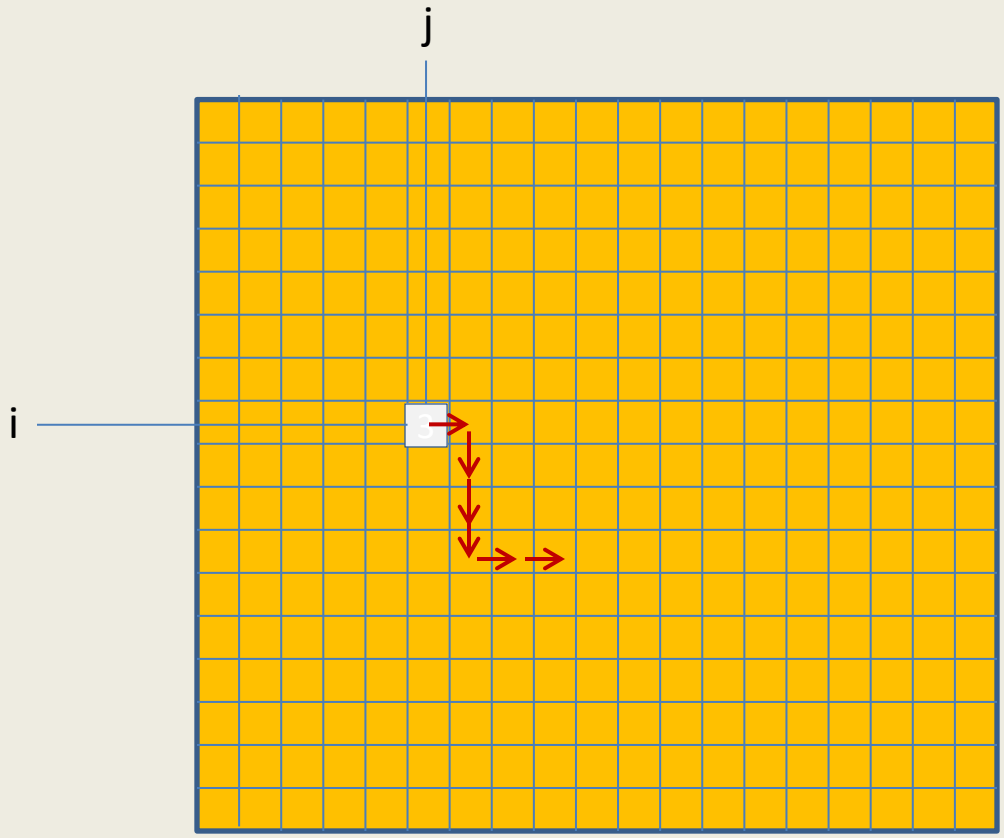
If you find a problem difficult,

➔ try to solve its simpler version, and then

➔ extend this solution to the original (difficult) version.

A new approach

Repeat : *if current entry is not local minima, explore the neighbor storing smaller value.*



A new approach

```
Explore()
{    Let c be any entry to start with;
     While(c is not a local minima)
     {
        c ← a neighbor of c storing smaller value
     }
     return c;
}
```

Question: What is the proof of correctness of **Explore** ?

Answer:

➔It suffices if we can prove that **While** loop eventually terminates.

➔Indeed, the loop terminates since **we never visit a cell twice**.

A new approach

Explore()

```
{    Let c be any entry to start with;
     While(c is not a local minima)
     {
        c ← a neighbor of c storing smaller value
     }
     return c;
}
```

Worst case time complexity : **O**(n^2)

How to apply this principle ?

First principle:
Do not discard **Explore**()

Second principle:
Simplify the problem

Local minima in an array

Theorem 2: A local minima in an array storing n distinct elements can be found in **O**(**log** n) time.

Homework:

- Design the algorithm stated in **Theorem 2**.
- Spend some time to extend this algorithm for the grid with running time= **O**(n).

Please come prepared in the next class ☺